Phase Randomisation: Numerical Study of Higher Cumulants Behaviour

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Abstract

For the purpose of testing for stationarity in a time series, a phase randomisation procedure is reviewed and modified, and applied to a wide range of time series models. These include linear stationary, linear non-stationary, non-linear stationary and non-linear non-stationary processes. Surrogate series are simulated using Standard and Rescaling methods. For all processes, the higher order central moments of the original series are preserved in the surrogate series using the Rescaling method whereas under the Standard approach only the even central moments are preserved. The density of higher cumulant estimates obtained under the Rescaling method exhibits unimodality when the process is stationary and multimodality otherwise. The primary aim is to develop a suite of diagnostic tests in order to assess the convergence of Markov Chain Monte Carlo algorithms. Applications of the method as a convergence diagnostic test of Markov Chain Monte Carlo are also discussed.

Keywords: higher cumulants; Markov Chain Monte Carlo; non-linear time series; stationarity ; surrogate series.

1 Introduction

1.1 Motivation

Simulation is now a standard tool in statistical analysis but, as with all numerical methods, demands careful attention to stability, convergence and other behaviour.

As an example, Markov Chain Monte Carlo (MCMC) methods have revolutionised Bayesian statistics, enabling evaluation of complex distributions and thus facilitating careful modelling in a very wide range of disciplines. An important consideration in the implementation of these methods, however, is whether the chain converges in some sense to the target distribution and, if so, how quickly. The output of a MCMC simulation exemplifies many situations in which simulated chains may be treated as a time series.

In this paper properties of such chains are investigated using the recently developed device used in the analysis of dynamical systems output, that of *phase randomisation.* This procedure consists of taking the Fourier transform of a given series, replacing the phase with a value sampled uniformly on $(0, 2\pi)$, and back-transforming to render a so-called *surrogate series.* Variously known as a method of surrogate data (Theiler *et al.* 1992), phase scrambling (Davison and Hinkley, 1997), or Fourier bootsrap (Braun and Kulperger, 1997), this method is commonly used to asses non-linearity in a time series (Theiler *et al.* 1992), or nonstationarity (Timmer, 1998), under the null hypothesis of a linear, Gaussian, stationary stochastic process.

Since the randomisation technique leaves amplitudes unaltered, second order structure is preserved in the surrogate series. Due to the requirement of symmetry among the array of angles used in the Fast Fourier Transform, it is far from trivial to determine which other features of the original distribution, if any, are preserved under phase randomisation.

In the long-term, the aim of phase randomisation in testing for stationarity of time series will be two-fold: firstly, estimators of functionals of distributions which we show to be fixed under phase randomisation will be found for the original and for replicated surrogate series and their concurrence quantified; and secondly, where non-stationarity is established, estimators of a nominated function will be found and it will be required to prove a Central Limit Theorem for each estimator so computed. The contribution of this paper is to address the first aim through empirical study. The second aim has been discussed in our work in Nur *et al* (2000a) and Nur *et al* (2000b). In Nur *et al* (2000a), the normality of third cumulant estimates obtained by phase randomisation is evaluated using traditional tests such as Kolmogorov-Smirnov and Shapiro and Wilk tests. Under stationary assumption, it is shown that the distribution is asymptotically normal. Furthermore, in Nur *et al* (2000b), by taking the third cumulant as the nominated function as mentioned in the second aim, we verify the conditions to prove the asymptotic distribution via Edgeworth expansion.

1.2 Overview of the surrogate method

The use of surrogate data to calibrate the statistical significance of any test statistic was suggested by Theiler *et al.* (1992). Theiler and Prichard (1992) compared two distinct approaches, namely *typical realizations* and *constrained realizations.* The typical realizations approach can be very powerful for the computation of confidence intervals provided the model equations can be extracted successfully whereas the constrained realizations method is more suitable for the purpose of hypothesis testing. Schreiber (1998) suggested a new method of creating artificial time sequences that fulfil given constraints but are random otherwise, which is fulfilled by minimizing a suitable cost function using *simulated annealing.* Timmer (1998) investigated the power of surrogate data for testing non-stationarity. The simulation studies reported in his paper indicate that surrogate data testing for linear, stochastic and Gaussian stationary processes is powerful against a violation of the assumption of stationarity.

Chan (1994) considered the validity of the method of surrogate data from the viewpoint of whether or not the nominal false rejection rate of a test statistic calibrated by the method of surrogate data approximately equals the true false rejection rate. He also derived some asymptotic properties of the method of surrogate data. Davison and Hinkley (1997) described some resampling schemes proposed for time series, one of which is *phase scrambling.* Braun and Kulperger (1993) studied the Fourier bootstrap as an adaptation of the surrogate data method of Theiler *et al.* (1992). The Fourier bootstrap is most suitable for stationary Gaussian sequences and performs reasonably well in some long-range dependence cases.

1.3 Organisation of Paper

In Section 2.1, we review the statistical properties of the surrogate data using phase randomisation, presented the algorithm in Section 2.2 and briefly explain the extension of the algorithm in this paper in Section 2.3. Section 3.1 presents the models and statistics used in the simulation results. The numerical results of higher cumulants' behaviour of some stationary and non-stationary, either linear or non-linear, using phase randomisation is discussed in Section 3.2. Finally discussion of current methods are presented in Section 4.1 and its implementation in particular contexts in Section 4.2. Appendix A, B, C and D contains all the numerical results, figures and plots discussed in Section 3.

2 Development of the method

2.1 The surrogate method

Suppose $\{X_t\}$ is a time series and $\mathbf{X_N} = (X_1, X_2, ..., X_N)^T$ is a data set. Let

$$E(X_t) = \mu, \quad \gamma(k) = \gamma_k = E(X_t - \mu)(X_{t+k} - \mu)$$

be the expectation and the autocovariances of $\{X_t\}$ respectively. Assume that the spectral density function, h, exists so that

$$\gamma_k = \int_{-\pi}^{\pi} \exp(ikw) h(w) dw, \qquad k = ..., -1, 0, 1, ...,$$

$$h(w) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k \exp(-ikw), \qquad -\pi \ \leq w \leq \ \pi.$$

Given $\mathbf{X}_N$, the Discrete Fourier Transform (DFT) is defined by

$$\psi(w) = \psi_{\mathbf{X}_N}(w) = \frac{1}{\sqrt{2\pi N}} \sum_{t=1}^{N} X_t \exp(-iwt), \qquad -\pi \leq w \leq \pi.$$

The modified periodogram is an estimator of h, i.e., $\hat{h}(w) = I(w)$ and is given by

$$I(\mathbf{X}, w) = I(\mathbf{X}_N, w) = I(w) = \mid \psi_{\mathbf{X}_n}(w) \mid^2 = \frac{1}{2\pi} \sum_{k=-(N-1)}^{N-1} r'_k \exp(-ikw)$$

where $r'_k = N^{-1} \sum_{t=1}^{N-k} X_t X_{t+k}$, is the k-th lag uncentred sample autocovariance.

The DFT is often only computed at a set of equally distributed angular frequencies: $w_j = 2\pi j/N$. Let $\psi_N = (\psi(w_1), ..., \psi(w_N))^T$. Hence $\mathbf{X}_N$ can be recovered from the DFT of ψ_N as

$$X_t = \sqrt{\frac{2\pi}{N}} \sum_{j=1}^{N} \psi(w_j) \exp(ijw_t), \quad t = 1, 2, ..., N. \tag{2.1}$$

Written in the polar form,

$$\psi(w_j) = \sqrt{I(w_j)} \exp(i\theta_j)$$

where $\sqrt{I(w_j)}$ is the amplitude and θ_j is the phase.

Thus when N is odd,

$$X_t = \bar{X} + \sqrt{\frac{2\pi}{N}} \sum_{j=1}^{(N-1)/2} 2\sqrt{I(w_j)} \cos(w_t j + \theta_j),$$

and when N is even,

$$X_t = \bar{X} + \sqrt{\frac{2\pi}{N}} \sum_{j=1}^{(N-2)/2} 2\sqrt{I(w_j)} \cos(w_t j + \theta_j) + \sqrt{\frac{2\pi}{N} I(\mathbf{X}, w_{n/2})} \cos(\pi t + \theta_{N/2}).$$

The method of surrogate data generates fictitious data $\mathbf{Y} = \mathbf{Y}_N = (Y_1, ..., Y_N)^T$ that preserves the observed sample mean and periodogram

$$Y_t = \bar{X} + \sqrt{\frac{2\pi}{N}} \sum_{j=1}^{m} 2\sqrt{I(w_j)} \cos(w_t j + \theta_j), \ \forall 1 \leq t \leq N, \ N = 2m+1, \tag{2.2}$$

where $\theta_1, ..., \theta_m$ are iid $U[0, 2\pi]$. If $N = 2(m+1)$, then

$$Y_t = \bar{X} + \sqrt{\frac{2\pi}{N}} \sum_{j=1}^{m} 2\sqrt{I(w_j)} \cos(w_t j + \theta_j) + \sqrt{\frac{2\pi}{N} I(\mathbf{X}, w_{n/2})} \cos(\pi t + \theta_{N/2}),$$

where $\theta_1, ..., \theta_m$ are iid $U[0, 2\pi]$ and independent of $\theta_{N/2}$ which is equal to 0 or π with probability 0.5 each.

By construction, the surrogate data $\mathbf{Y}$ preserve the observed sample mean and periodogram, that is

$$\bar{Y} = \bar{X}, \quad I(\mathbf{Y}, w_j) = I(\mathbf{X}, w_j), \quad \forall j = 1, 2, ..., N \textit{ with probability } 1.$$

As a consequence, $\mathbf{Y}$ preserves the sample circular auto-covariances that is

$$\sum_{t=1}^{N} (Y_t - \bar{Y})(Y_{t+k} - \bar{Y})/N = \sum_{t=1}^{N} (X_t - \bar{X})(X_{t+k} - \bar{X})/N$$

where $Y_{t+N} = Y_t$ and $X_{t+N} = X_t$ for all $t \geq 0$.

Let $S = S_N = S(X_N)$ denote the sample mean $\bar{X}$ and the periodogram values $I(\mathbf{X}, w_j), j = 1, ..., N$. Assume that $S = s$ is fixed and $\mathbf{Y}$ is generated according to (2.2). Assume that $N = 2m+1$ is odd, $p \geq q$ be two positive integers. Then

$$E(Y_t) = \bar{X},$$

$$Cov(Y_p, Y_q) = E\left[(Y_p - \bar{X})(Y_q - \bar{X})\right] - (\bar{X})^2 = r_{p-q,c},$$

as $\theta_1, ..., \theta_m$ are iid $U[0, 2\pi]$, $E\left[cos(w_t j + \theta_j)\right] = 0$ and $Var\left[cos(w_t j + \theta_j)\right] = \frac{1}{2}$.

Continuing for higher cumulants, we have the following properties

- $C(Y_p, Y_q, Y_r) = E\left[(Y_p - \bar{X})(Y_q - \bar{X})(Y_r - \bar{X})\right] = 0$

- Odd cumulants of the surrogate series are zero
- Even cumulants of the surrogate series are non-zero

This implies that the surrogate series have a highly symmetric joint distribution.

Under the assumption of stationarity and Gaussianity, it is proved by Braun and Kulperger (1997) that with a certain sufficient condition, the Fourier bootstrap is asymptotically normal by Braun and Kulperger (1997, Theorem 2). Furthermore they noted that there is a large class of processes for which this result holds. Their example, that some long-memory processes which satisfy an ergodicity condition, is also proved by Chan (1997, Theorem 2.6) under the assumptions of that the process is linear, stationary and Gaussian and that the spectral density function is bounded away from zero and infinity.

2.2 Algorithms

In the following, we state two algorithms for phase randomisation detailed by Theiler *et al.* (1992) and Davison and Hinkley (1997) respectively. The first algorithm is based on the null hypothesis that the data come from a linear stochastic process. The assumption in this algorithm is that there is no non-linearity either in the dynamics or in the observation of the data. As the surrogate series produced by the first algorithm will have a highly symmetric joint distribution, Davison and Hinkley (1997) suggested the second algorithm below when the original data have an asymmetric marginal distribution.

Standard algorithm

(1) Input the original data into an array $x[t],\ t = 1, 2, ..., N$.

(2) Compute the Discrete Fourier Transform: $z[t] = DFT(x[t])$. Note that $z[t]$ has real and imaginary components.

(3) Randomize the phases: $z'[t] = z[t]\exp(i\phi[t])$, where $\phi[t]$ is uniformly distributed between 0 and 2π.

(4) Symmetrize the phases such that

$$\Re z''[t] = \Re\left(z'[t] + z'[N+1-t]\right)/2$$

$$\Im z''[t] = \Im\left(z'[t] - z'[N+1-t]\right)/2$$

where $\Re$ and $\Im$ are the real and imaginary parts of a complex number respectively.

(5) Invert the DFT: $x'[t] = DFT^{-1}(z''[t])$.

(6) The resulting time series $x'[t]$ is the surrogate data.

Rescaling Surrogate algorithm

(1) Input the original data into an array $x[t],\ t = 1, 2, ..., N$.

(2) Let $y_t = \Phi^{-1}\left\{\frac{r_t}{(n+1)}\right\}$ where r_j is the rank of x_t among the original series $x_1, ..., x_N$.

(3) Apply Standard algorithm to $y_1, ..., y_N$, giving $Y_1^*, ..., Y_N^*$.

(4) Set the surrogate series $X_t^* = x_{(r_t')}$, where r_t' is the rank of Y_t^* among $Y_1^*, ..., Y_N^*$.

2.3 Extension

2.3.1 Testing Hypothesis

The null hypothesis of surrogate data testing for linearity is that the data were generated by a linear, stochastic, Gaussian stationary process, including a possible non-linear observation function. A rejection of this hypothesis does not necessarily mean that the data come from a chaotic, that is, some kind of stationary, non-linear deterministic process. They might also originate from a non-linear, stochastic, stationary; a non-linear stochastic, non-stationary or even simply from a linear, stochastic, non-stationary process.

Let Q_D be the statistic computed for the original data, Q_{H_i} be the statistic computed for the i-th surrogate data generated under the null hypothesis, and μ_H and σ_H denote the mean and standard deviation of the distribution of Q_H. Define

$$S = \frac{\mid Q_D - \mu_H \mid}{\sigma_H}.$$

If the distribution of the statistic is Gaussian then the p-value is given by $p = erfc(S/\sqrt{2})$.

2.3.2 Timmer's work

Timmer applied the method of surrogate data to cyclostationary processes. He chose cyclostationary processes because these processes allow a simple way to find a parametric violation of the null hypothesis. He showed two violations of stationarity in the frame of cyclostationary processes : firstly, increasing the amplitude of modulation and secondly, increasing the period of modulation. He used the correlation dimension as a non-linear feature. Hence, Q_D is the correlation dimension of the original data, the mean of the distribution of this feature for the surrogate data by μ_H, and its variance by σ_H^2. The simulation studies reported that starting from amplitude modulation of 0.3 for the first violation and from period modulation of 1.5 for the second violation, the null hypothesis is clearly rejected at the 5% level of siginificance. Hence Timmer (1998) has shown that the rejection of null hypothesis is caused by a nonstationarity of the processes.

2.3.3 Our work

In order to see whether the higher cumulants of original series are preserved under phase randomisation, we computed the estimate of higher cumulants of original and surrogate series respectively. We expect that the higher cumulant estimates are preserved under the null hypothesis (linear, Gaussian and stationary). Furthermore, we would like to investigate the behaviour of the higher cumulant estimates when the process is linear and non-stationary; non-linear and stationary or non-stationary.

One thousand surrogate series were constructed based on each of the time series models listed in Table 3.1. Estimates of the functions in Table 2.1 below were calculated for each original and surrogate series with length of 200. Hence, Q_D is the higher cumulants estimates of the original data, the mean of the distribution of this feature for the surrogate data by μ_H, and its variance by σ_H^2. The results of numerical study is presented in the next section with detailed in the appendix.

Table 2.1. Higher cumulants functions and estimates at lags $k = 1, ..., 20$

Functions	Mathematical Forms	Estimates
Higher central moments	$E[(X_t - \mu)]^r, \quad r = 2, ..., 7$	$\frac{1}{N}\sum_{t=1}^{N}(X_t - \bar{X})^r$
Higher order cumulants	$E[\prod_{j=1}^{r}(X_{t+k+j} - \mu)], \ r = 2, ..., 6$	$\frac{1}{N}\sum_{t=1}^{N-k-r}[\prod_{j=1}^{r}(X_{t+k+j} - \bar{X})]$
Odd/Even − cross cumulants	$E[(X_t - \mu)^r(X_{t+k} - \mu)^r], \quad r = 1, 2.$	$\frac{1}{N}\sum_{t=1}^{N-k-r}[(X_t - \mu)^r(X_{t+k} - \mu)^r]$

3 Numerical Study

3.1 Models

The time series models are listed in Table 3.1. For non-linear and stationary processes, we generated the models with weak and strong non-linearity properties. As an example, for the bilinear stationary series, the first model (3.3) looks similar to the linear models as the non-linearity term is small, whereas the second bilinear stationary model (3.4) has a strong non-linear property. In all models, $\varepsilon_t \sim N(0,1)$.

Table 3.1. Models that simulated in the paper

Name	Models	Model No
AR(1) stationary	$X_t = 0.4X_{t-1} + \varepsilon_t$	(3.1)
Random Walk	$X_t = X_{t-1} + \varepsilon_t$	(3.2)
Bilinear stationary	$X_t = 0.1X_{t-1} + 0.2X_{t-1}\varepsilon_{t-1} + \varepsilon_t$	(3.3)
Bilinear stationary	$X_t = 0.1X_{t-1} + 0.7X_{t-1}\varepsilon_{t-1} + \varepsilon_t$	(3.4)
Bilinear non − stationary	$X_t = 0.6X_{t-1} + 0.8X_{t-1}\varepsilon_{t-1} + \varepsilon_t$	(3.5)
GARCH stationary	$X_t = 1.0 + 0.1X_{t-1} + 0.2X_{t-1}\varepsilon^2_{t-1} + \varepsilon_t$	(3.6)
GARCH stationary	$X_t = 1.0 + 0.1X_{t-1} + 0.8X_{t-1}\varepsilon^2_{t-1} + \varepsilon_t$	(3.7)
GARCH non − stationary	$X_t = 1.0 + 0.51X_{t-1} + 0.5X_{t-1}\varepsilon^2_{t-1} + \varepsilon_t$	(3.8)
ThresholdAR stationary	$X_t = \begin{cases} -0.7X_{t-1} + \varepsilon_t & \text{if } X_{t-1} \geq 0 \\ -0.9X_{t-1} + \varepsilon_t & \text{if } X_{t-1} < 0 \end{cases}$	(3.9)
ThresholdAR stationary	$X_t = \begin{cases} 0.1X_{t-1} + \varepsilon_t & \text{if } X_{t-1} \geq 0 \\ -10.0X_{t-1} + \varepsilon_t & \text{if } X_{t-1} < 0 \end{cases}$	(3.10)
ThresholdAR non − stationary	$X_t = \begin{cases} -0.9X_{t-1} + \varepsilon_t & \text{if } X_{t-1} \geq 0 \\ -1.1111111X_{t-1} + \varepsilon_t & \text{if } X_{t-1} < 0 \end{cases}$	(3.11)
ThresholdAR non − stationary	$X_t = \begin{cases} X_{t-1} + \varepsilon_t & \text{if } X_{t-1} \geq 0 \\ 2.0 - 0.3X_{t-1} + \varepsilon_t & \text{if } X_{t-1} < 0 \end{cases}$	(3.12)

3.2 Results

We present the results in the following tables analysing the timeplots, estimates of higher moments, estimates of higher cumulants and density estimates of each method for each model in Table 3.1. The timeplots are depicted in Appendix A; the estimates of higher moments and cumulants are in Appendix B and C respectively and Appendix D contains the density plots of some higher cumulants estimates using the Rescaling method. In each table in appendix, the estimates of higher cumulants are calculated for some lags, that is, the estimates of higher moments are presented at lag 0. In the first row of each lag in each table, the estimates of higher cumulants of original series are presented, followed by the mean and standard deviation, denoted by () and [] brackets, respectively, of the estimates using the two methods. At the end of this subsection, the results for each method are summarised.

The results for each method are summarised as follows.

Standard Surrogate method

(1) For linear, Gaussian and stationary processes, higher moments and cumulants (including cross cumulants) of original processes are preserved.

(2) For non-linear and stationary processes, higher moments of original processes are preserved but some of higher cumulants and cross-cumulants of original processes are not preserved. The values of higher cumulants of the original are non-zero and small while those of surrogate are zero for the odd higher cumulants only. This implies that the method produces a more symmetric surrogate series than the original.

(3) For linear and non-stationary processes, the second and cross cumulants tend to be unpreserved as the lag increases. The odd moments/cumulants tend to be unpreserved showing low p- values. The values of higher cumulants are non-zero and large while for their surrogate the odd cumulants are around zero. Again, the method results in a more symmetry surrogate than the original.

(4) For non-linear and non-stationary processes, the higher moments are preserved whereas the higher cumulants of original processes are not preserved. The values of higher cumulants are non-zero and large while for their surrogate the odd cross cumulants are around zero (except GARCH).

(5) The smoothing densities of odd cumulants tend to be symmetric around zero while the smoothing densities of even cumulants tend to be skew with positive or negative values only near zero and multimodal. For non-stationary processes, the values are large, multimodal, and the modes of the smoothing density are shifted from zero.

Rescaling Surrogate method

(1) For linear, Gaussian and stationary processes, higher cumulants of original processes are preserved. The method is not appropriate for this class of model as most of the second cumulants are not preserved.

(2) For non-linear and stationary processes, the second cumulants tend to have low p-values but are still insignificant. The higher moments are preserved but this is not so for higher cumulants. Most of the cumulants of the surrogate are around zero which implies that the surrogate is symmetric, or even Gaussian. The method does not work well for the Threshold stationary process.

(3) For linear and non-stationary processes, the second cumulants tend to be unpreserved. The higher moments are preseved but higher cumulants (cross) can be significantly different from the original. The values of higher moments and cumulants of the original are non-zero and large, as are those of the surrogate. It shows that the original and surrogate are both non Gaussian.

(4) For non-linear and non-stationary processes, higher moments are preserved and the higher cumulants are very significantly different from the original. The values of higher moments and cumulants of the original and surrogate are non-zero and large. The odd cumulants of the surrogate are non-zero. It shows that the original and surrogate are both non Gaussian.

(5) From the smoothing density of higher cumulants of stationary processes in the Figures 5.3-5.5 in Appendix 5.4, odd and even cumulants are not significantly different from zero, can be skew, are unimodal with the mode around zero and have small scale. For non-stationary processes, the values have large scale, are multimodal or unimodal with long-tails and the modes or mode of the smoothing density are located away from zero.

4 Discussion

In the following, we summarise our discussion regarding the performance of phase randomisation with particular focus on the behaviour of the higher cumulants of current methods in Section 4.1. We also discuss the potential applications of these results in Section 4.2.

4.1 Performance of phase randomisation

The following overall conclusions are thus reached

(1) By the Standard surrogate method, if the original process is stationary with standard normal error density (Gaussian or approximately Gaussian), then the surrogate series will be approximately Gaussian. If the original process is non-stationary (non Gaussian) then the surrogate series is more symmetric than the original. The method works well for linear and weakly non-linear and stationary processes. For other models, the surrogate series will be more symmetric than the original.

(2) By the Rescaling method, if the original process is stationary with standard normal error density (Gaussian or approximately Gaussian), then the surrogate series will be approximately Gaussian. If the original process is non-stationary then the surrogate series will not be Gaussian. In general, if the process is stationary, the densities of the higher cumulants of the surrogate are unimodal while for non-stationary processes they are multimodal. The method works well for strongly non-linear time series models and non-stationary processes. For other models, the surrogate series will be more symmetric than the original.

Table 4.1. Conclusion on the behaviour of higher cumulants for stationary and non-stationary processes using the Standard Surrogate method

Process	Original	Surrogate
Stationary	Higher moments are non-zero	Higher moments are non-zero
	Higher cumulants around zero or small	Higher odd cumulants are around zero or small
	Cross cumulants are small	Cross cumulants are around zero (odd) or quite small (even)
Nonstationary	Higher moments are non-zero large	Higher moments are non-zero
	Higher cumulants are very large	Higher cumulants are quite large (even) but odd cumulants are zero
	Cross cumulants are nonzero	Cross cumulants are quite large (even), but odd cumulants are around zero

Table 4.2. Conclusion on the behaviour of higher cumulants for stationary and non-stationary processes using the Rescaling method

Process	Original	Surrogate
Stationary	Higher moments are non-zero	Higher moments are non-zero
	Higher cumulants around zero or small	Higher cumulants are around zero
	Cross cumulants are small	Cross cumulants are around zero (odd) or quite small (even)
Nonstationary	Higher moments are non-zero	Higher moments are non-zero
	Higher cumulants are very large	Higher cumulants are quite large
	Cross cumulants are large	Cross cumulants are quite large

As a result of these observations, we conclude that we can access the stationarity of a process by investigating at the higher cumulants estimates of the surrogate series constructed by the Rescaling method. In particular, the stationarity of an original process can be characterised from the density of higher cumulant estimates of their surrogates as follows:

(1) if the density of the higher cumulants estimates of surrogates are unimodal around zero with a small variance then the original process is linear stationary or weakly non-linear stationary,

(2) if the density of the higher cumulants estimates of surrogates are unimodal around zero with a quite large variance then the original process is strongly non-linear (e.g. Bilinear, Threshold AR) stationary,

(3) if the density estimates of the higher cumulants estimates are unimodal around zero with a bit tail and quite large variance then the original process is strongly non-linear (e.g. GARCH, Threshold) stationary.

Otherwise, if the density estimates of the higher cumulants estimates are multimodal or tend to be unimodal with a very long tail and nonzero mode then the original process is non-stationary.

4.2 Implementation for MCMC

As yet in unpublished technical report by D. Nur, K.L.Mengersen and R.C.Wolff (2000), the phase randomisation method is implemented to compare its performance as a convegence diagnostic test to other diagnostics tests that available in the literature (Cowles and Carlin, 1996 and Mengersen *et al.* 1999). This method is powerful to detect the unstability of a MCMC in the burn-in period, which can be undetected by other methods. This result can be generalised to detect the stability of a process in more general context or as a model checking tool.

We present the application of phase randomisation as a convergence diagnostic test compared to other tests in CODA software in the following example.

4.2.1 Rats example

In a study conducted by the CIBA-GEIGY company, the weights of 30 young rats in a control group were measured weekly for five weeks. The data was given in Gelfand *et al.* (1990), with weight measurements available for all five weeks.

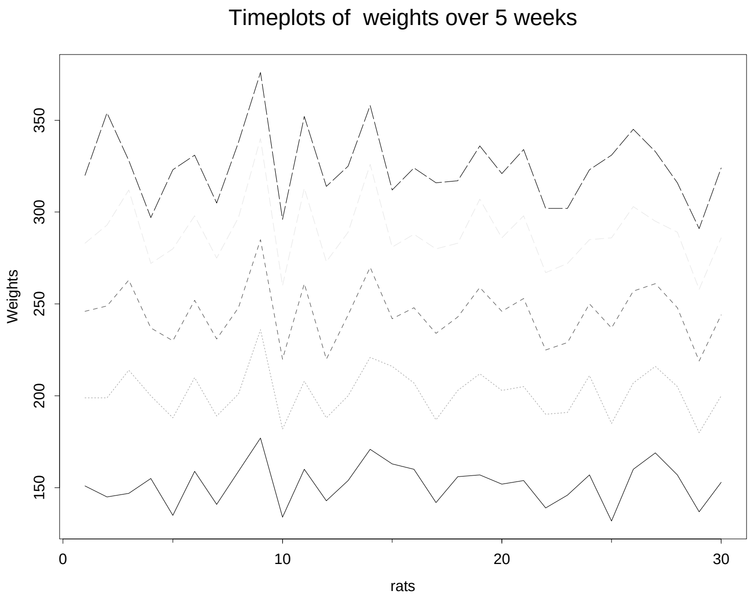

Figure 4.0. Timeplots of rat's weight over 5 weeks

For the time period considered, it is reasonable to assume individual straight-line growth curves. It is also assumed the homoscedastic normal measurements errors so that

$$Y_{ij} \sim Normal(\alpha_i + \beta_i x_{ij}, \sigma_c^2),\ i = 1, ..., k;\ j = 1, ..., n_i,$$

provides the full measurement model (with $k = 30$, $n_i = 5$ and x_{ij} denoting the age in days of the$i$th rat when measurement j was taken). The population structure is modeled as

$$\begin{pmatrix} \alpha_c \\ \beta_c \end{pmatrix} \sim Normal\left\{\begin{pmatrix} \alpha_c \\ \beta_c \end{pmatrix}, \sigma_c\right\}$$

assuming conditional independence throughout. A full Bayesian analysis now requires the specification of a prior for $\sigma_c^2, (\alpha_c, \beta_c)$, and σ_c. The following prior are assigned :

$$\alpha_c \sim Normal(0, 0.0001), \quad \beta_c \sim Normal(0, 0.0001),$$

$$\tau_c \sim Gamma(0.001, 0.001), \quad \tau_c \sim Gamma(0.001, 0.001), \quad \tau_c \sim Gamma(0.001, 0.001).$$

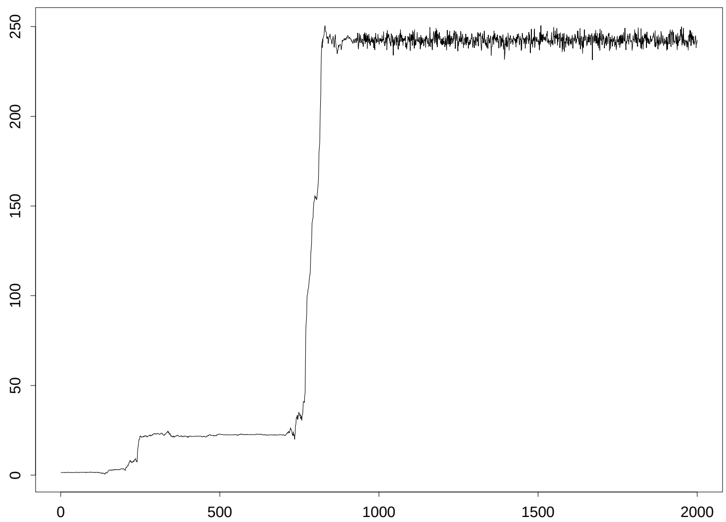

Figure 4.1. Rats example: MCMC's plot of parameter α_c, 2000 iterations

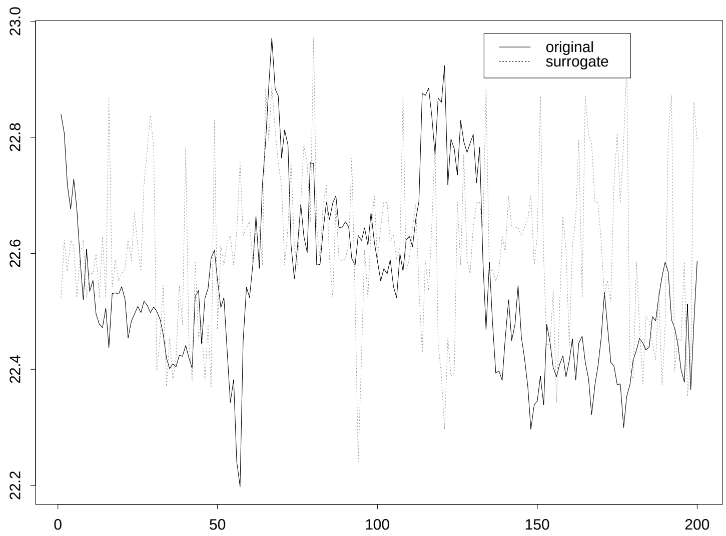

Figure 4.2. Rats example: The 500-700th of Figure 4.1. Unstable.

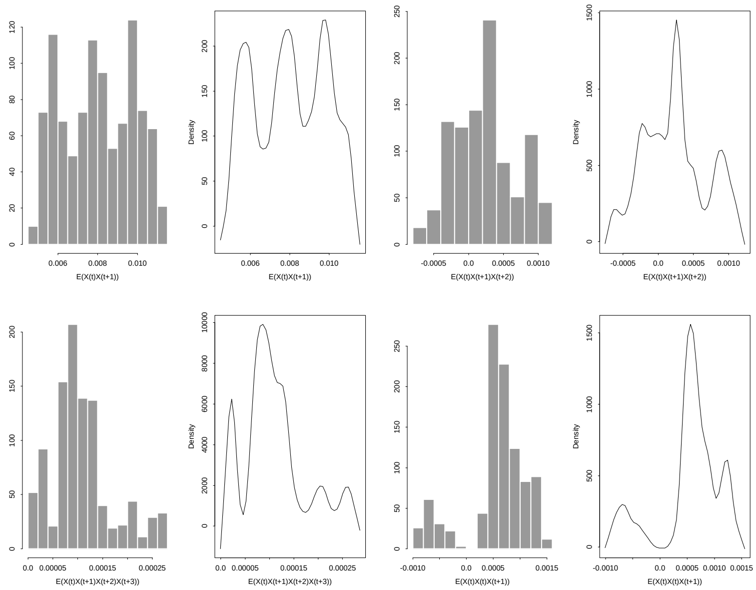

Figure.4.3. Rats example: The smoothing density of higher cumulants of Figure 4.2.

In this example, we analyse the model starting from unusual initial conditions

$$\alpha_c = 0.1,\ \beta_c = 0,\ \tau_\alpha = 1000,\ \tau_\beta = 1,\ \tau_c = 1$$

to compare the results of phase randomisation method to other methods. Starting from these unusual initial conditions, the posterior distribution of parameters are being stable averagely after 1000 iterations. We apply these methods to the early part which is non-stationary and at the end part which is stationary. All the methods in CODA except Raftery-Lewis are calculated using the default values in CODA. For Raftery-Lewis, as the number of observation is 200, the default changes are : the precision is 0.02 with probability 0.90.

The timeplot of the first 2000 iterations of this example is exhibited in Figure 4.1 for which the convergence will be obtained after 1000 iterations. Figure 4.2 shows the timeplot of 500-700th iterations of Figure 4.1 which is similar to the time plot of random walk. Figure 4.3 show the smoothing densities of higher cumulant estimates of the series in Figure 4.2 which are multimodal. This reflects the non-stationary properties.

The conclusion from Table 4.3 for parameter α, the higher cumulants estimates of surrogate series are multimodal for the series in Figure 4.2, which shows the nonstationarity of MCMC. The similar results are obtained by using Raftery-Lewis, Heidelberger-Welch and autocorrelations tests. In contrary, BUGS and Geweke's test confirm the stationarity of MCMC.

Table 4.3. Results of tests for series in Figure 4.2

Tests	*Values*	*Results*
BUGS	-1.91	Passed
Geweke	1.66	Passed
Raftery − Lewis	1.48	Failed
Heidelberger − Welch	1.62	Failed
Autocorrelations	High at lags 1,5,10	Failed
Phase randomisation	Multimodal	Failed

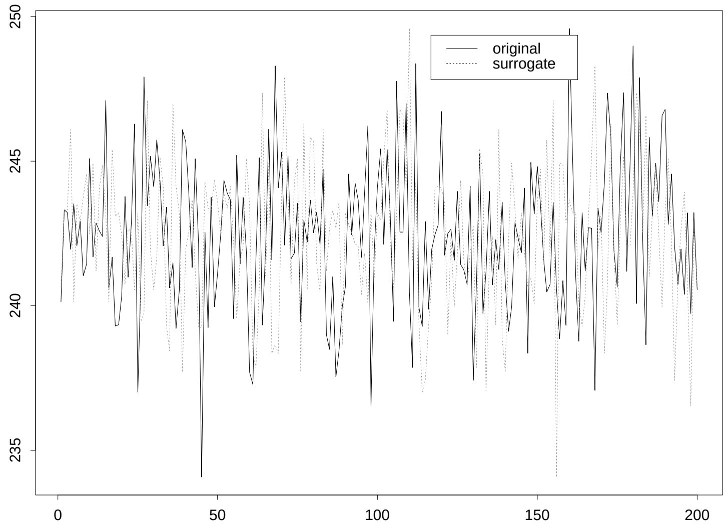

Figure 4.4. Rats example: The 1000-1200th of Figure 4.1. Stable.

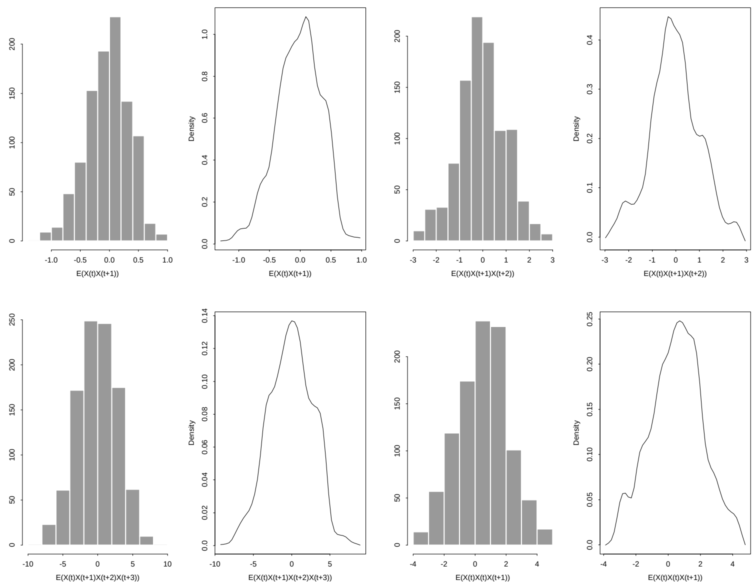

Figure 4.5. Rats example: The smoothing density of higher cumulants of Figure 4.4.

Furthermore, the phase randomisation is applied to the 1000-1200th iterations as the timeplot in Figure 4.4. Figure 4.4 shows the similarity to the time plot of bilinear stationary or Threshold AR stationary. The smoothing densities of higher cumulant estimates in Figure 4.5 show the unimodality around zero with a quite large variance to confirm the stationarity. All tests confirm the stationary of the series in Figures 4.5 (see Table 4.4) after discarding a few initial observations.

Table 4.4. Results of tests for series in Figure 4.4

Tests	*Values*	*Results*
BUGS	0.924	Passed
Geweke	-1.12	Passed
Raftery − Lewis	0.97	Passed, 2 burn-in
Heidelberger − Welch	0.16	Passed(200)
Autocorrelations	Low at all lags	Passed
Phase randomisation	Unimodal around zero, a quite large var.	Passed

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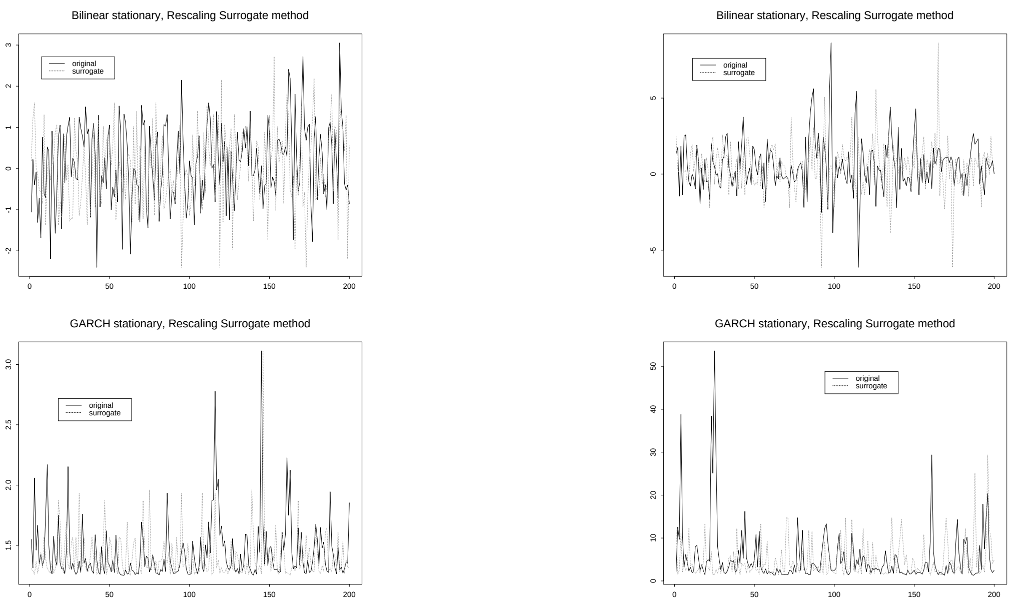
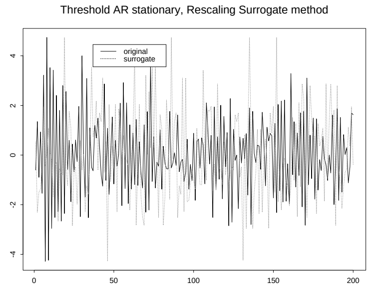
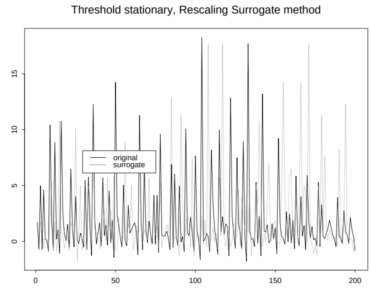
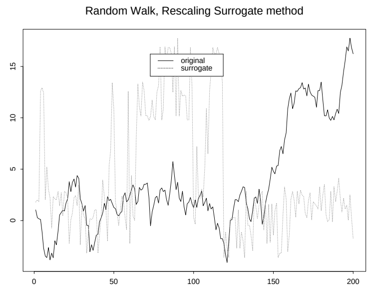
A Timeplots

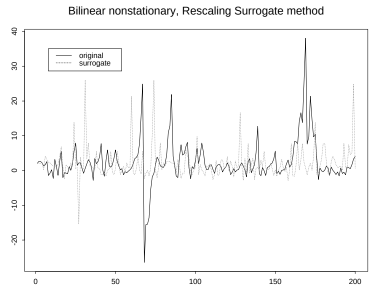
Figure 5.2. Timeplots of models (3.2)-(3.11) respectively using Rescaling method

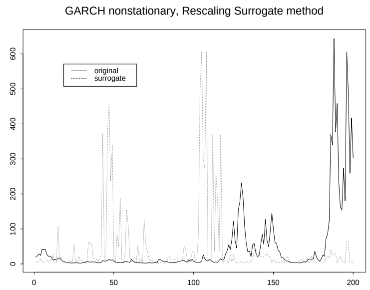


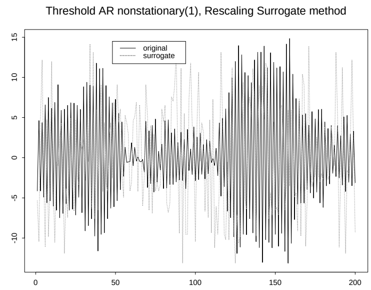

Threshold AR stationary, Rescaling Surrogate method
original
surrogate


Threshold stationary, Rescaling Surrogate method
original
surrogate


Random Walk, Rescaling Surrogate method
original
surrogate


Bilinear nonstationary, Rescaling Surrogate method
original
surrogate


GARCH nonstationary, Rescaling Surrogate method
original
surrogate


Threshold AR nonstationary(1), Rescaling Surrogate method
original
surrogate

B Tables of higher moments estimates

Table 5.1. The higher moments estimates for stationary processes in Table 3.1

Models and methods	Third	Fourth	Fifth	Sixth
AR(1)(3.1)	**-0.2230 x**10^{-1}	**0.3429x**10^{1}	**-0.3526**	**0.1379x**10^{2}
Standard	*(-0.1000x*10^{-4}*)* *[0.7410x*10^{-1}*]*	*(0.5084x*10^{1}*)* *[0.4941x*10^{1}*]*	*(0.9000x*10^{-4}*)* *[0.1629x*10^{1}*]*	*(0.3347x*10^{2}*)* *[0.3830x*10^{2}*]*
Rescaling	(-0.4200x10^{-3}) [0.1637]	(0.3232x10^{1}) [0.6044]	(-0.5432) [0.1285x10^{1}]	(0.1285x10^{2}) [0.3005x10^{1}]
Bilinear(3.3)	**0.3400x**10^{-2}	**0.2395x**10^{1}	**0.6059**	**0.1107x**10^{2}
Standard	*(-0.1100x*10^{-5}*)* *[0.1140x*10^{-1}*]*	*(0.3630x*10^{1}*)* *[0.3527x*10^{1}*]*	*(0.2005x*10^{-3}*)* *[0.1797x*10^{1}*]*	*(0.2832x*10^{2}*)* *[0.3240x*10^{2}*]*
Rescaling	(0.1092) [0.1921]	(0.2413x10^{1}) [0.6327]	(0.1287x10^{1}) [0.1918x10^{1}]	(0.1178x10^{2}) [0.5221x10^{1}]
GARCH (3.6)	**0.4863x**10^{-1}	**0.6531x**10^{-1}	**0.9383x**10^{-1}	**0.1442**
Standard	*(0.3800x*10^{-5}*)* *[0.7750x*10^{-1}*]*	*(0.9892x*10^{-1}*)* *[0.9610x*10^{-1}*]*	*(0.2200x*10^{-4}*)* *[0.2693]*	*(0.3714)* *[0.4249]*
Rescaling	(0.5414x10^{-1}) [0.2190x10^{-1}]	(0.7250x10^{-1}) [0.3360x10^{-1}]	(0.1031) [0.5190x10^{-1}]	(0.1554) [0.8210x10^{-1}]
Threshold (3.9)	**0.6304**	**0.1944 x**10^{2}	**0.1149 x**10^{2}	**0.2299x**10^{3}
Standard	*(0.1960x*10^{-2}*)* *[0.1070x*10^{1}*]*	*(0.2924x*10^{2}*)* *[0.2845x*10^{2}*]*	*(0.7378x*10^{-1}*)* *[0.3549x*10^{2}*]*	*(0.5800x*10^{3}*)* *[0.6648x*10^{3}*]*
Rescaling	(0.6609) [0.5808]	(0.2178x10^{2}) [0.1079x10^{2}]	(0.1292x10^{2}) [0.1332x10^{2}]	(0.2864x10^{3}) [0.1983x10^{3}]

Table 5.2. The higher moments estimates for non-stationary processes in Table 3.1

Models and methods	Third	Fourth	Fifth	Sixth
Random Walk (3.2)	**0.1232x**10^{3}	**0.1915x**10^{4}	**0.1652x**10^{5}	**0.2135x**10^{6}
Standard	*(0.6720x*10^{-1}*)* *[0.1510x*10^{3}*]*	*(0.2519x*10^{4}*)* *[0.2449x*10^{4}*]*	*(0.4467x*10^{1}*)* *[0.3016x*10^{5}*]*	*(0.3626x*10^{6}*)* *[0.4151x*10^{6}*]*
Rescaling	(0.1257x10^{3}) [0.6283x10^{2}]	(0.2203x10^{4}) [0.1265x10^{4}]	(0.1859x10^{5}) [0.9680x10^{4}]	(0.2559x10^{6}) [0.1408x10^{6}]
Bilinear (3.5)	**0.2884x**10^{3}	**0.1801x**10^{5}	**0.2656x**10^{6}	**0.1567x**10^{8}
Standard	*(-0.1600x*10^{-1}*)* *[0.3306x*10^{3}*]*	*(0.2717x*10^{5}*)* *[0.2640x*10^{5}*]*	*(-0.3202x*10^{2}*)* *[0.4022x*10^{6}*]*	*(0.3935x*10^{8}*)* *[0.4502x*10^{8}*]*
Rescaling	(0.1895x10^{3}) [0.2013x10^{3}]	(0.1509x10^{5}) [0.7399x10^{4}]	(0.1405x10^{6}) [0.2316x10^{6}]	(0.1181x10^{8}) [0.8064x10^{7}]
GARCH (3.8)	**0.3709x**10^{7}	**0.1761x**10^{10}	**0.8869x**10^{12}	**0.4686x**10^{15}
Standard	*(-0.2553x*10^{4}*)* *[0.5769x*10^{7}*]*	*(0.2636x*10^{10}*)* *[0.2563x*10^{10}*]*	*(-0.3684x*10^{9}*)* *[0.2530x*10^{13}*]*	*(0.1217 x*10^{16}*)* *[0.1390 x*10^{16}*]*
Rescaling	(0.3398x10^{7}) [0.2196x10^{7}]	(0.1521x10^{10}) [0.1108x10^{10}]	(0.6992x10^{12}) [0.5335x10^{12}]	(0.3439x10^{15}) [0.2726x10^{15}]
Threshold (3.12)	**0.4008x**10^{2}	**0.4765 x**10^{4}	**0.9461 x**10^{4}	**0.6171 x**10^{6}
Standard	*(0.1011)* *[0.6323x*10^{2}*]*	*(0.7141x*10^{4}*)* *[0.6950x*10^{4}*]*	*(0.4497x*10^{2}*)* *[0.2673x*10^{5}*]*	*(0.1541x*10^{7}*)* *[0.1767x*10^{7}*]*
Rescaling	(0.5485x10^{2}) [0.2391x10^{2}]	(0.4497x10^{4}) [0.9775x10^{3}]	(0.1262x10^{5}) [0.5711x10^{4}]	(0.5767x10^{6}) [0.1524x10^{6}]

[1]

[1] The bold are the values for original series, the values inside () and [] brackets are the estimates of mean and standard deviation of surrogates.

C Tables of higher cumulants estimates

Table 5.3. The higher cumulants estimates for stationary processes in Table 3.1

Models	Lags	Third	Fourth	Fifth	Sixth	Cross
AR(1)	**1**	**0.7750x**10^{-1}	**0.3821**	**0.6960x**10^{-1}	**0.2245**	**-0.3550x**10^{-1}
(3.1)	*Standard*	*(0.2x*10^{-4}*)*	*(0.6515)*	*(-0.6x*10^{-5}*)*	*(0.5589)*	*(-0.21x*10^{-4}*)*
		[0.1041]	*[0.6331]*	*[0.9870x*10^{-1}*]*	*[0.6396]*	*[0.1084]*
	Rescaling	(0.9800x10^{-2})	(-0.4400x10^{-2})	(0.4190x10^{-1})	(0.3760x10^{-2})	(-0.1620x10^{-1})
		[0.6890x10^{-1}]	[0.8290x10^{-1}]	[0.9920x10^{-1}]	[0.1082]	[0.1031]
	10	**0.2360x**10^{-1}	**-0.1613**	**-0.1229**	**-0.2641**	**-0.1546**
	Standard	*(-0.7900x*10^{-5}*)*	*(-0.2386)*	*(0.3900x*10^{-4}*)*	*(-0.7428)*	*(0.4830x*10^{-2}*)*
		[0.4080x10^{-1}*]*	*[0.2319]*	*[0.6827]*	*[0.8501]*	*[0.1632]*
	Rescaling	(0.5050x10^{-4})	(0.4240x10^{-1})	(0.1603x10^{-1})	(-0.4390x10^{-2})	(0.4680x10^{-2})
		[0.7542x10^{-1}]	[0.7543x10^{-1}]	[0.7930x10^{-1}]	[0.8910x10^{-1}]	[0.9870x10^{-1}]
	20	**0.1838**	**0.1207**	**0.2537**	**0.1098**	**-0.252x**10^{-1}
	Standard	*(0.75x*10^{-4}*)*	*(0.2823)*	*(-0.5x*10^{-4}*)*	*(0.6276)*	*(0.14x*10^{-4}*)*
		[0.3897]	*[0.2743]*	*[0.8819]*	*[0.7182]*	*[0.7363x*10^{-1}*]*
	Rescaling	(-0.3142x10^{-1})	(0.1020x10^{-1})	(-0.175x10^{-1})	(-0.137x10^{-1})	(0.1756x10^{-1})
		[0.679x10^{-1}]	[0.802x10^{-1}]	[0.829x10^{-1}]	[0.1016]	[0.1078]
Bilinear	**1**	**-0.2701x**10^{-1}	**-0.4463x**10^{-1}	**0.2570x**10^{-1}	**-0.2670x**10^{-1}	**0.3289**
(3.3)	*Standard*	*(0.209x*10^{-5}*)*	*(-0.173x*10^{-1}*)*	*(0.16x*10^{-4}*)*	*(-0.748x*10^{-1}*)*	*(-0.48x*10^{-4}*)*
		[0.212x10^{-1}*]*	*[0.168x*10^{-1}*]*	*[0.1044]*	*[0.856x*10^{-1}*]*	*[0.4881]*
	Rescaling	(-0.359x10^{-2})	(-0.107x10^{-1})	(0.77x10^{-2})	(0.974x10^{-2})	(0.2003x10^{-1})
		[0.458x10^{-1}]	[0.538x10^{-1}]	[0.472x10^{-1}]	[0.387x10^{-1}]	[0.557x10^{-1}]
	10	**-0.3133x**10^{-1}	**-0.1935 x**10^{-1}	**-0.558x**10^{-2}	**0.307 x**10^{-2}	**-0.713x**10^{-2}
	Standard	*(0.107x*10^{-5}*)*	*(-0.542x*10^{-1}*)*	*(-0.246x*10^{-5}*)*	*(0.3866x*10^{-1}*)*	*(0.901x*10^{-5}*)*
		[0.109x10^{-1}*]*	*[0.527x*10^{-1}*]*	*[0.2203x*10^{-1}*]*	*[0.4424x*10^{-1}*]*	*[0.916x*10^{-1}*]*
	Rescaling	(0.2386x10^{-1})	(-0.846x10^{-2})	(-0.1639x10^{-1})	(-0.104x10^{-2})	(-0.928x10^{-3})
		[0.629x10^{-1}]	[0.469x10^{-1}]	[0.421x10^{-1}]	[0.412x10^{-1}]	[0.644x10^{-1}]
	20	**-0.792x**10^{-3}	**0.361x**10^{-2}	**-0.4612x**10^{-1}	**-0.1621x**10^{-1}	**0.381x**10^{-2}
	Standard	*(0.137x*10^{-5}*)*	*(0.1252x*10^{-1}*)*	*(-0.119x*10^{-4}*)*	*(-0.366x*10^{-1}*)*	*(-0.599x*10^{-6}*)*
		[0.139x10^{-1}*]*	*[0.122x*10^{-1}*]*	*[0.1067]*	*[0.419x*10^{-1}*]*	*[0.6089x*10^{-2}*]*
	Rescaling	(-0.66x10^{-2})	(-0.769x10^{-2})	(0.1516x10^{-1})	(-0.195x10^{-3})	(0.1148x10^{-1})
		[0.569x10^{-1}]	[0.5345x10^{-1}]	[0.401x10^{-1}]	[0.411x10^{-1}]	[0.739x10^{-1}]

[2]

[2] The bold are the values for original series, the values inside () and [] brackets are the estimates of mean and standard deviation of surrogates.

Table 5.3(continued). The higher cumulants estimates for stationary processes in Table 3.1

Models	Lags	Third	Fourth	Fifth	Sixth	Cross
GARCH	**1**	**0.555x**10^{-2}	**0.212x**10^{-2}	**0.616x**10^{-3}	**0.94x**10^{-4}	**0.1211x**10^{-1}
(3.6)	*Standard*	*(0.46x*10^{-6}*)*	*(0.322x*10^{-2}*)*	*(0.14x*10^{-6}*)*	*(0.25x*10^{-3}*)*	*(0.81x*10^{-6}*)*
		[0.947x10^{-2}*]*	*[0.312x*10^{-2}*]*	*[0.181x*10^{-2}*]*	*[0.29x*10^{-3}*]*	*[0.1661x*10^{-1}*]*
	Rescaling	(0.193x10^{-3})	(-0.187x10^{-4})	(0.89x10^{-6})	(0.72x10^{-6})	(-0.524x10^{-3})
		[0.869x10^{-3}]	[0.15x10^{-3}]	[0.25x10^{-4}]	[0.45x10^{-5}]	[0.448x10^{-2}]
	10	**-0.272x**10^{-3}	**-0.632x**10^{-3}	**-0.278x**10^{-3}	**-0.802x**10^{-4}	**-0.512x**10^{-2}
	Standard	*(0.406x*10^{-7}*)*	*(-0.869x*10^{-3}*)*	*(-0.503x*10^{-7}*)*	*(-0.14x*10^{-3}*)*	*(-0.39x*10^{-6}*)*
		[0.83x10^{-3}*]*	*[0.84x*10^{-3}*]*	*[0.63x*10^{-3}*]*	*[0.16x*10^{-3}*]*	*[0.802x*10^{-2}*]*
	Rescaling	(-0.703x10^{-3})	(-0.24x10^{-5})	(-0.19x10^{-6})	(-0.16x10^{-5})	(-0.1698x10^{-2})
		[0.74x10^{-3}]	[0.17x10^{-3}]	[0.22x10^{-4}]	[0.56x10^{-5}]	[0.285x10^{-2}]
	20	**-0.842x**10^{-3}	**-0.259x**10^{-4}	**-0.59 x**10^{-4}	**-0.32x**10^{-4}	**-0.194x**10^{-2}
	Standard	*(-0.38x*10^{-7}*)*	*(-0.49x*10^{-4}*)*	*(-0.13x*10^{-7}*)*	*(-0.95x*10^{-4}*)*	*(-0.19x*10^{-6}*)*
		[0.79x10^{-3}*]*	*[0.49x*10^{-4}*]*	*[0.17x*10^{-3}*]*	*[0.11x*10^{-3}*]*	*[0.387x*10^{-2}*]*
	Rescaling	(0.618x10^{-3})	(0.18x10^{-4})	(-0.97x10^{-5})	(0.24x10^{-5})	(-0.189x10^{-3})
		[0.104x10^{-2}]	[0.19x10^{-3}]	[0.26x10^{-4}]	[0.56x10^{-5}]	[0.386x10^{-2}]
Threshold	**1**	**-0.503x**10^{-1}	**0.9457x**10^{1}	**0.8952**	**-0.6463x**10^{2}	**0.1863**
(3.9)	*Standard*	*(-0.11x*10^{-3}*)*	*(0.1418x*10^{2}*)*	*(0.403x*10^{-2}*)*	*(-0.1614x*10^{3}*)*	*(0.384x*10^{-3}*)*
		[0.575x10^{-1}*]*	*[0.1379x*10^{2}*]*	*[0.1939x*10^{1}*]*	*[0.1849x*10^{3}*]*	*[0.2097]*
	Rescaling	(-0.1237)	(0.3611x10^{-1})	(0.1983)	(-0.6928x10^{-1})	(0.5933x10^{-1})
		[0.2870]	[0.4039]	[0.6078]	[0.8238]	[0.3940]
	10	**0.1495**	**-0.2411x**10^{1}	**-0.7640**	**0.6289x**10^{1}	**0.3547**
	Standard	*(0.43x*10^{-4}*)*	*(-0.4196x*10^{1}*)*	*(-0.96x*10^{-3}*)*	*(0.2495x*10^{2}*)*	*(0.105x*10^{-2}*)*
		[0.236x10^{-1}*]*	*[0.4083x*10^{1}*]*	*[0.4617]*	*[0.2860x*10^{2}*]*	*[0.5734]*
	Rescaling	(-0.791x10^{-2})	(-0.1298)	(0.1857)	(-0.4749)	(0.2432x10^{-1})
		[0.3510]	[0.5754]	[0.7816]	[0.1919x10^{1}]	[0.3566]
	20	**0.1940**	**-0.4534**	**-0.3886x**10^{-1}	**0.6485x**10^{-1}	**0.2409**
	Standard	*(0.525x*10^{-3}*)*	*(0.2945)*	*(-0.172x*10^{-2}*)*	*(-0.3136x*10^{2}*)*	*(0.893x*10^{-3}*)*
		[0.2866]	*[0.2866]*	*[0.8278]*	*[0.3594x*10^{2}*]*	*[0.4880]*
	Rescaling	(0.406x10^{-1})	(-0.1360)	(-0.516x10^{-1})	(-0.1217)	(-0.3316)
		[0.2719]	[0.4363]	[0.8020]	[0.1358x10^{1}]	[0.6323]

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[3] The bold are the values for original series, the values inside () and [] brackets are the estimates of mean and standard deviation of surrogates.

Table 5.4. The higher cumulants estimates for non-stationary processes in Table 3.1

Models	Lags	Third	Fourth	Fifth	Sixth	Cross
Random Walk	**1**	**0.1137**x10^3	**0.1586**x10^4	**0.1181**x10^5	**0.1225**x10^6	**0.1171**x10^3
(3.2)	*Standard*	*(0.6404x10^{-1})*	*(0.2164x10^4)*	*(0.3536x10^1)*	*(0.2435x10^6)*	*(0.6396x10^{-1})*
		[0.1438x10^3]	*[0.2104x10^4]*	*[0.2388x10^5]*	*[0.2787x10^6]*	*[0.1436x10^3]*
	Rescaling	(0.1032x10^3)	(0.1305x10^4)	(0.1105x10^5)	(0.1138x10^6)	(0.1129x10^3)
		[0.5737x10^2]	[0.8841x10^3]	[0.7458x10^4]	[0.8481x10^5]	[0.5944x10^2]
	10	**0.8950**x10^2	**0.9823**x10^3	**0.8362**x10^4	**0.8003** x10^5	**0.7289**x10^2
	Standard	*(0.764x10^{-1})*	*(0.1025x10^4)*	*(0.3492x10^1)*	*(0.1218x10^6)*	*(0.226x10^{-1})*
		[0.1716x10^3]	*[0.9962x10^3]*	*[0.2357x10^5]*	*[0.1395x10^6]*	*[0.5083x10^2]*
	Rescaling	(0.2488x10^2)	(0.3275x10^3)	(0.2048x10^4)	(0.2025x10^5)	(0.3130x10^2)
		[0.4885x10^2]	[0.5917x10^3]	[0.5648x10^4]	[0.5579x10^5]	[0.3493x10^2]
	20	**0.6194**x10^2	**0.7817**x10^3	**0.6606**x10^4	**0.7023**x10^5	**0.6152**x10^2
	Standard	*(0.5680x10^{-1})*	*(0.7178x10^3)*	*(0.2336x10^1)*	*(0.9159x10^3)*	*(0.1703x10^{-1})*
		[0.1276x10^3]	*[0.6979x10^3]*	*[0.1568x10^5]*	*[0.1049x10^6]*	*[0.3826x10^2]*
	Rescaling	(0.1489x10^2)	(0.1799x10^3)	(0.1157x10^4)	(0.9285x10^4)	(0.2244x10^2)
		[0.4342x10^2]	[0.3681x10^3]	[0.3471x10^4]	[0.2906x10^5]	[0.2578x10^2]
Bilinear	**1**	**0.8169**x10^2	**0.2262**x10^4	**0.2003**x10^5	**0.2543**x10^6	**0.2682**x10^2
(3.5)	*Standard*	*(-0.599x10^{-2})*	*(0.3256x10^4)*	*(-0.4519x10^1)*	*(0.5765x10^6)*	*(-0.89x10^{-3})*
		[0.1237x10^3]	*[0.3163x10^4]*	*[0.5677x10^5]*	*[0.6595x10^6]*	*[0.1852x10^2]*
	Rescaling	(0.6722x10^1)	(-0.7177x10^1)	(0.2511x10^2)	(-0.1597x10^3)	(-0.938x10^{-1})
		[0.2006x10^2]	[0.7071x10^2]	[0.1270x10^3]	[0.3811x10^3]	[0.4321x10^2]
	10	**-0.5682**x10^1	**0.3943**x10^2	**-0.2590**x10^4	**-0.4007**x10^5	**-0.8764**x10^1
	Standard	*(-0.817x10^{-3})*	*(0.193x10^3)*	*(0.3405)*	*(-0.8161x10^5)*	*(0.1311x10^{-2})*
		[0.1690x10^2]	*[0.1875x10^3]*	*[0.4277x10^4]*	*[0.9337x10^5]*	*[0.2709x10^2]*
	Rescaling	(-0.5155x10^1)	(-0.1242x10^2)	(-0.6416x10^2)	(0.4079x10^3)	(-0.2815x10^1)
		[0.1044x10^2]	[0.6063x10^2]	[0.1715x10^3]	[0.1126x10^4]	[0.3397x10^2]
	20	**0.3083**x10^1	**-0.1929**x10^3	**-0.2154**x10^4	**-0.1571**x10^5	**-0.4843**x10^2
	Standard	*(-0.63x10^{-3})*	*(-0.2209x10^3)*	*(0.4187)*	*(-0.2862x10^5)*	*(0.334x10^{-2})*
		[0.1304x10^2]	*[0.2146x10^3]*	*[0.5259x10^4]*	*[0.3274x10^5]*	*[0.6906x10^2]*
	Rescaling	(-0.4141x10^1)	(-0.6399x10^1)	(-0.5952)	(-0.2312x10^3)	(0.1204x10^2)
		[0.7924x10^1]	[0.7283x10^2]	[0.1093x10^3]	[0.6073x10^3]	[0.3978x10^2]

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[4] The bold are the values for original series, the values inside () and [] brackets are the estimates of mean and standard deviation of surrogates.

Table 5.4(continued). The higher cumulants estimates for non-stationary processes in Table 3.1

Models	Lags	Third	Fourth	Fifth	Sixth	Cross
GARCH	**1**	**0.2001x**10^7	**0.5594x**10^9	**0.1301x**10^{12}	**0.2198x**10^{14}	**0.2674x**10^7
(3.6)	*Standard*	*(-0.1375x*10^4*)*	*(0.8298x*10^9*)*	*(-0.5285x*10^8*)*	*(0.5744.7x*10^{14}*)*	*(-0.1814x*10^4*)*
		[0.3107x10^7*]*	*[0.8068x*10^9*]*	*[0.3629x*10^{12}*]*	*[0.6576x*10^{14}*]*	*[0.4099x*10^7*]*
	Rescaling	(0.7739x10^6)	(0.1756x10^9)	(0.3002x10^{11})	(0.5309x10^{13})	(0.1369x10^7)
		[0.4802x10^6]	[0.1515x10^9]	[0.3109x10^{11}]	[0.7266x10^{13}]	[0.9088x10^6]
	10	**0.7974x**10^6	**0.1758x**10^9	**0.3925x**10^{11}	**0.7480x**10^{11}	**0.1238x**10^7
	Standard	*(-0.5478x*10^3*)*	*(0.2325x*10^9*)*	*(-0.1340x*10^8*)*	*(0.1399x*10^{14}*)*	*(-0.7515x*10^3*)*
		[0.1238x10^7*]*	*[0.2260x*10^9*]*	*[0.9204x*10^{11}*]*	*[0.1603x*10^{14}*]*	*[0.1698x*10^7*]*
	Rescaling	(0.1662x10^5)	(0.1404x10^8)	(0.3788x10^9)	(0.5146x10^{11})	(0.1895x10^6)
		[0.1665x10^6]	[0.5694x10^8]	[0.6416x10^{10}]	[0.1511x10^{13}]	[0.6530x10^6]
	20	**-0.2799x**10^6	**-0.7319x**10^8	**-0.1999x**10^{11}	**-0.4697x**10^{13}	**0.2121x**10^5
	Standard	*(0.1054x*10^3*)*	*(-0.8530x*10^8*)*	*(0.6408x*10^7*)*	*(-0.9904x*10^{13}*)*	*(0.2529x*10^3*)*
		[0.2382x10^6*]*	*[0.8293x*10^8*]*	*[0.4401x*10^{11}*]*	*[0.1134x*10^{14}*]*	*[0.5714x*10^6*]*
	Rescaling	(-0.1336x10^6)	(-0.3259x10^8)	(-0.7259x10^{10})	(-0.1572x10^{13})	(-0.1819x10^6)
		[0.1647x10^6]	[0.3265x10^8]	[0.8800x10^{10}]	[0.1930x10^{13}]	[0.2692x10^6]
Threshold	**1**	**-0.1215x**10^2	**0.4394x**10^4	**0.1571x**10^4	**-0.4824x**10^6	**-0.1205x**10^2
(3.12)	*Standard*	*(-0.2869x*10^{-1}*)*	*(0.6574x*10^4*)*	*(0.7197x*10^1*)*	*(-0.1204x*10^6*)*	*(-0.333x*10^{-1}*)*
		[0.1795x10^2*]*	*[0.6399x*10^4*]*	*[0.4278x*10^4*]*	*[0.1380x*10^7*]*	*[0.2087x*10^2*]*
	Rescaling	(-0.5392)	(0.6484x10^2)	(-0.9068x10^2)	(-0.2988x10^4)	(0.5606x10^1)
		[0.2389x10^2]	[0.2005x10^3]	[0.1747x10^4]	[0.1255x10^5]	[0.2599x10^2]
	10	**-0.8264x**10^1	**-0.3670x**10^4	**0.1188x**10^4	**0.4035x**10^6	**0.3036x**10^2
	Standard	*(-0.2429x*10^{-1}*)*	*(-0.5441x*10^4*)*	*(0.6464x*10^1*)*	*(0.1004x*10^7*)*	*(0.7778x*10^{-1}*)*
		[0.1519x10^2*]*	*[0.5295x*10^4*]*	*[0.3842x*10^4*]*	*[0.1151x*10^7*]*	*[0.4864x*10^2*]*
	Rescaling	(-0.7370x10^1)	(-0.7980x10^2)	(-0.1993x10^1)	(-0.1780x10^4)	(0.7273x10^1)
		[0.2171x10^2]	[0.1703x10^3]	[0.1339x10^4]	[0.1137x10^5]	[0.2304x10^2]
	20	**-0.4994x**10^1	**-0.2904x**10^4	**0.4996x**10^3	**0.3109x**10^6	**0.2198x**10^2
	Standard	*(-0.1798x*10^{-1}*)*	*(-0.4222x*10^4*)*	*(0.3486x*10^1*)*	*(0.7677x*10^6*)*	*(0.5612x*10^{-1}*)*
		[0.1124x10^2*]*	*[0.4109x*10^4*]*	*[0.2072x*10^4*]*	*[0.8801x*10^6*]*	*[0.3509x*10^2*]*
	Rescaling	(-0.4239x10^1)	(0.2323x10^2)	(0.2689x10^3)	(0.9160x10^3)	(0.1702x10^1)
		[0.2339x10^2]	[0.1695x10^3]	[0.1302x10^4]	[0.1240x10^5]	[0.2073x10^2]

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[5] The bold are the values for original series, the values inside () and [] brackets are the estimates of mean and standard deviation of surrogates.

D Smoothing density figures by Rescaling method

Figure 5.3. Smoothing density of Bilinear stationary (3.3) (left) and Bilinear stationary (3.4) (right)

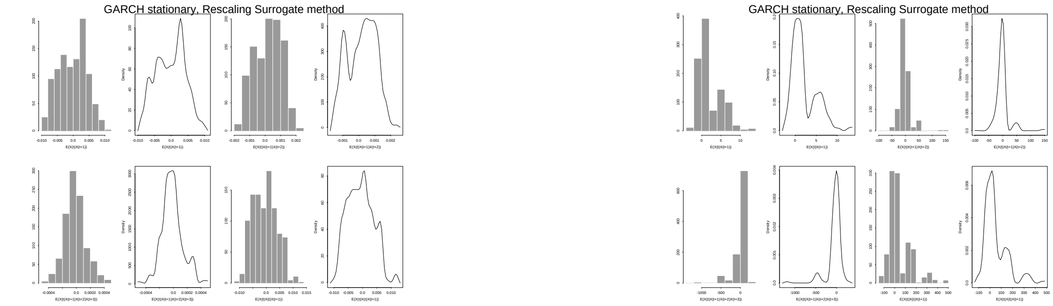

Figure 5.4. Smoothing density of GARCH stationary (3.6) (left) and GARCH stationary (3.7) (right)

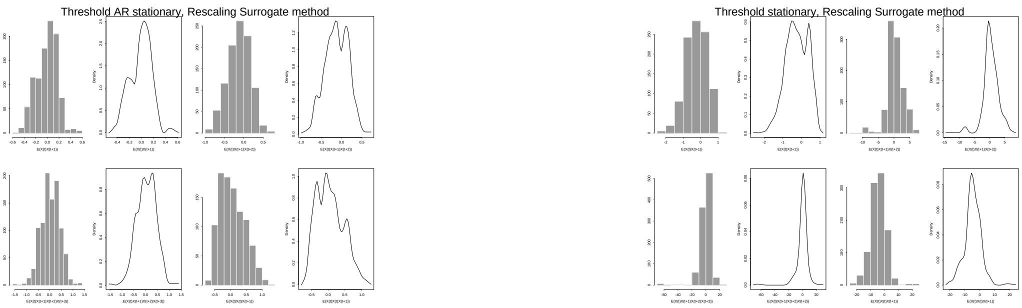

Figure 5.5. Smoothing density of Threshold AR stationary (3.9) (left) Threshold AR stationary (3.10) (right)

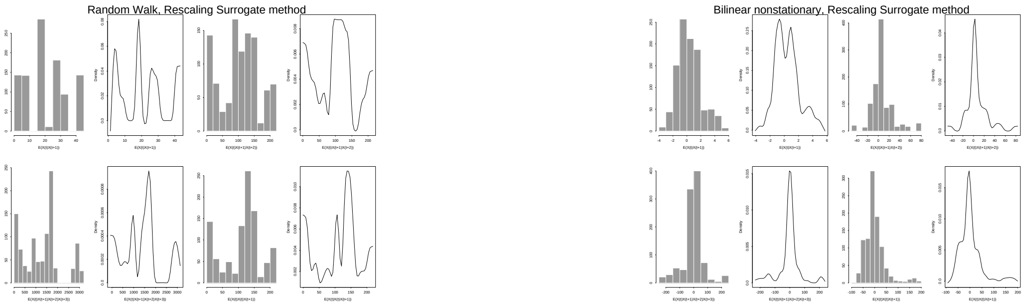

Figure 5.6. Smoothing density of Random Walk (3.2) (left) and Bilinear non-stationary (3.5) (right)

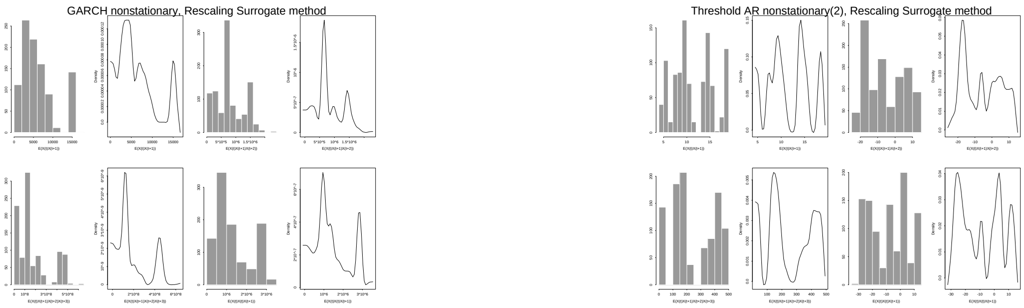

Figure 5.7. Smoothing density of GARCH non-stationary (3.8) (left) and Threshold non-stationary (3.12) (right)